A COMPARATIVE ANALYSIS OF POTENTIAL-BASED AND NON-POTENTIAL-BASED COHESIVE MODELS FOR SIMULATING LOADING AND UNLOADING IN SLIDING ELASTIC LAMINATES

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ABSTRACT

When materials break, the process is incredibly complex. To predict and analyze these failures, engineers and scientists use computational tools called Cohesive Zone Models (CZMs)1. These models fall into two main camps: potential-based models, which use a single energy function to describe the fracture process, and non-potential-based models, which offer more freedom to describe complex behaviors, especially when a material is repeatedly loaded and unloaded2. This study takes a close look at both approaches, comparing them in the context of laminated materials sliding against each other3. We've developed a unified mathematical framework that allows us to build both types of models from the ground up, showing how to incorporate the effects of loading and unloading in a natural way, even when the fracture properties change with direction4. Our analysis, backed by clear numerical examples, reveals major differences in how these models predict stiffness, energy loss, and permanent damage5. We show that potential-based models, while elegant, have serious limitations and can produce unrealistic results under the complex conditions of mixed-mode unloading6. In contrast, the non-potential-based approach proves to be a robust and reliable tool that works well in all situations7. These findings help clarify which model to choose for a given problem and offer critical insights for accurately predicting failure in advanced composite materials.

Keywords: Cohesive Zone Model (CZM); Potential-Based Models; Non-Potential-Based Models; Variational Methods; Fracture Mechanics; Laminated Composites; Delamination; Anisotropy; Mixed-Mode Fracture; Cyclic Loading; Hysteresis; Quasi-static Evolution; Finite Element Method (FEM).

Introduction

1.1. Background and Motivation

For over a century, scientists and engineers have sought to understand and predict why and how materials break10. This quest began with A.A. Griffith's groundbreaking work [14], which showed that fracture could be understood as a balance of energy11. For a long time, the field was dominated by a theory known as Linear Elastic Fracture Mechanics (LEFM), which works wonderfully for very brittle materials like glass, where the breakage is clean and happens almost instantly at the tip of a crack12. However, many modern materials, from plastics to advanced composites, aren't so simple13. They stretch, deform, and accumulate damage in a messy, nonlinear zone ahead of a crack, a reality that LEFM and its assumption of a perfectly sharp crack tip can't quite handle14. To bridge this gap, the Cohesive Zone Model (CZM) was born15. Independently envisioned by Barenblatt [3] and Dugdale [9], the CZM offers a more realistic picture16. Instead of an infinitely sharp crack tip, it imagines a small "process zone" where the material is gradually tearing apart17. Within this zone, special "cohesive" forces try to hold the material together18. The relationship describing how these forces weaken as the material separates is called the traction-separation law (TSL), and it forms the heart of the model19. This elegant idea gets rid of the problematic singularities of LEFM and gives us a single, unified way to think about how fractures start and how they grow20.

1.2. Two Competing Philosophies: Variational vs. Non-Variational Models

In recent years, a powerful mathematical approach known as the variational method has become a popular way to study fracture [5, 11]21. The core idea is to reframe fracture as a process of energy minimization—a crack will form and grow in a way that minimizes the total energy of the system22. Models built on this philosophy are called potential-based because the forces that drive the fracture are all derived from a single energy function, or "potential"23. This is an attractive approach because it guarantees that the model will obey the laws of thermodynamics, ensuring energy is properly conserved and dissipated [24]24. The well-known Park-Paulino-Roesler (PPR) model is a prime example of this philosophy in action [22]25.

However, this mathematical elegance comes at a price26. Forcing all behavior to stem from one energy function is a major constraint27. Real-world materials, when pushed and pulled repeatedly, exhibit all sorts of complex behaviors—their stiffness changes, they don't always return to their original shape, and they lose energy in ways that are hard to describe with a simple potential28. To capture this rich reality, researchers have developed

non-potential-based models29. These models don't rely on an overarching energy function30. Instead, they are defined directly by the equations of force balance, giving modelers the freedom to write down separate rules for what happens during loading, unloading, and reloading31. This flexibility allows for a much more accurate description of what we actually see in experiments, as highlighted in the work of McGarry et al. [17]32.

1.3. The Unsolved Puzzle: Handling Complex Loading and Anisotropy

One of the biggest challenges in damage modeling is correctly capturing how a material behaves differently when it's being loaded versus when it's being unloaded33. The standard trick is to give the model a kind of memory by introducing a "history variable" that keeps track of the most extreme deformation the material has ever experienced34. This allows the model to know whether it's in a new, damaging loading phase or a less-damaging unloading phase35. This concept is essential for tackling problems like metal fatigue [8] and is often used alongside models for contact and friction [25]36. This puzzle becomes even harder when dealing with materials whose properties change with direction—a feature known as

anisotropy37. This is common in modern laminated composites and thin films, where the material might be much stronger at resisting being pulled apart than it is at resisting a shearing or sliding force [15, 16]38. While some recent models have tried to include unloading effects [4, 23], they have mostly been limited to simpler, isotropic materials39. A clear, analytical way to build non-potential models that can handle both anisotropy and history-dependent unloading has remained an open question40. This is a critical gap, because to ensure the safety and reliability of advanced structures like hybrid laminates [1, 2], we need models that can predict their behavior under the complex, multi-directional loads they experience in the real world41.

1.4. What This Paper Aims to Achieve

This paper tackles these challenges head-on42. We present a clear and unified comparison of potential-based and non-potential-based cohesive models, focusing on the specific problem of sliding elastic laminates43. Our main contributions are:

- A Unified Framework: We develop a rigorous mathematical language that can describe both types of models44. Within this framework, we introduce a natural, step-by-step method to build sophisticated models that handle loading and unloading, starting from simpler, pure-loading laws45.
- A Direct Comparison: We systematically compare the two approaches, using both theory and practical examples to show where potential-based models fall short, especially in mixed-mode unloading scenarios46. We demonstrate that these models are only reliable under very specific conditions, while the non-potential framework works well across the board47.
- Mathematical Proofs: We apply our models to the sliding laminate problem and provide rigorous mathematical proofs that our solutions are well-posed and exist, extending the work of previous studies [6, 19, 20]48.

By clarifying the pros and cons of each approach, we hope to provide researchers and engineers with clear guidance on how to choose the right tool for the job when it comes to analyzing failure in today's advanced materials49.

2. Models and Methods

2.1. Describing the System: The Mechanical and Mathematical Setup

Let's imagine our composite material as a system of two elastic layers, Omega_1 and Omega_2, bonded together along a flat interface, Gamma_c50. We'll work in the world of linearized elasticity, which is a good approximation as long as the deformations are small [7]51. The state of our system is fully described by the displacement field,

u=(u_1,u_2), which tells us how every point in each layer has moved52. The total energy of the system comes from two sources: the elastic energy stored in the bulk of the layers and the surface energy of the cohesive interface53. The elastic energy, E, is calculated as:

E(u)=i=1∑221∫ΩCi(x)e(ui(x)):e(ui(x))dx

Here, e(u_i) is the strain (how much the material is stretched or sheared), and C_i(x) is the stiffness tensor, which describes how resistant each layer is to deformation54. We assume the standard properties for C_i that ensure the problem is well-behaved [7]55. The action happens at the interface, Gamma_c. The behavior here depends on the

displacement slip, δ(x) = u_1(x) − u_2(x), which is the relative movement between the two layers56. To handle cases where the interface responds differently to being pulled apart versus being sheared (anisotropy), we define a vector of cohesive variables, $g(\delta)$57. This function takes the slip vector and translates it into a set of scalar numbers that measure the separation in different modes58. To give our model a memory of past damage, we introduce a history variable, $\gamma(t,x)$59. Think of γ as a high-water mark; each of its components, γ_l, records the maximum value that the corresponding separation measure, g_l, has ever reached up to time t60. This ensures that damage is irreversible—the high-water mark can only go up, never down61.

2.2. The Potential-Based (Variational) Approach

In this approach, the interface behavior is governed entirely by a cohesive energy functional, K62:

$$K(\delta,\gamma)=\int_\Omega \Phi(x,g(\delta(x)),\gamma(x))dA$$

The function Φ is the cohesive energy density. It depends on the current separation, $y=g(\delta)$, and the damage history, $z=\gamma$63. The total energy of the system, F, is simply the sum of the elastic and cohesive energies64:

$$F(t,u,\gamma)=E(u)+K(u1-u2,\gamma)$$

To find out how the system evolves over time, we look for what's called a (generalized) energetic solution [18]65. This is a path, $(u(t),\gamma(t))$, that satisfies two key principles at all times66:

- Global Stability (GS): The system is always in a state of minimum energy67. For a given amount of historical damage, $\gamma(t)$, the actual displacement, $u(t)$, is the one that minimizes the total energy F68. The damage itself must also respect its history, meaning $\gamma(t,x)$ must be greater than or equal to the current separation measure $g(\delta(t,x))$69.
- Energy Balance (EB): Energy is conserved. The change in the system's total energy over time must equal the work done on it by external forces70.

In essence, this approach turns the physics problem into a mathematical problem of minimization71. The forces at the interface, the cohesive tractions, emerge naturally as the derivatives of the energy density, $T = \nabla_y \Phi$72.

2.3. A Recipe for Building the Potential, Φ

Defining a physically realistic energy density Φ that works for both loading and unloading is tricky73. We propose a constructive recipe. We start with a simpler "pure loading" density, $\Psi(y_1,y_2)$, which we assume is already known74. Then, for any given damage history, $z=(z_1,z_2)$, we divide the space of possible current separations, (y_1,y_2), into four zones: pure loading (both y_1,y_2 are increasing), pure unloading (both are decreasing), and two mixed zones75. The full density $\Phi(y,z)$ is then built piece by piece76. In the pure loading zone, it's just equal to the original $\Psi(y)$77. In the zones involving unloading, we define Φ to be quadratic in the unloading variable78. The coefficients of this quadratic are chosen carefully to ensure that the energy and forces are continuous as the system transitions between loading and unloading79. This recipe gives us a unique potential that results in tractions that decrease linearly during unloading—a behavior that is commonly assumed in damage models and often observed in reality80.

2.4. The Non-Potential-Based (Non-Variational) Approach

This approach takes a more direct route. It sidesteps the idea of a global energy function and instead defines the system's state directly through the equations of force balance (or, more formally, the principle of virtual work)81. At any moment in time, the system is in equilibrium if the internal forces from elastic strain balance the cohesive forces at the interface82. In its weak form, this is83:

$$\int_\Omega C1e(u1):e(\phi1)dx+\int_\Omega C2e(u2):e(\phi2)dx=\int_{\Gamma c} T\cdot(\phi1-\phi2)dA$$

This equation must hold for any valid "virtual" displacement φ84. The key difference here is that the cohesive traction, T, is not tied to a potential85. It's defined directly by a constitutive law, $T = T(x,g(\delta),\gamma)$, where T is the cohesive tension vector86. This gives us the freedom to design much more complex, path-dependent rules for how the interface behaves87. A (generalized) equilibrium solution is simply a path, $(u(t),\gamma(t))$, that satisfies this equilibrium equation at all times, while respecting the irreversibility of the history variable γ88.

2.5. A Recipe for Building the Tension

Just as with the potential, we can build a complete loading-unloading tension law, $T(y,z)$, from a simpler pure-loading law, $S(y)$89. The construction is again piecewise across the four loading/unloading zones. In the pure loading zone, T is just $S(y)$90. In the unloading zones, we define the tension to decrease linearly from its peak value back to zero91. For instance, in the pure unloading zone, the tension in mode i would be $T_i(y,z)=S_i(z)\cdot(y_i/z_i)$92. What should we choose for the loading tension, $S(y)$? A very effective and robust choice, which neatly avoids the problems of potential-based models, is to take the positive part of the gradient of a loading potential Ψ93:

$$S(y)=(\nabla y\Psi(y))+=(\max(0,\partial y1\Psi),\max(0,\partial y2\Psi))$$

This simple but powerful choice prevents the model from ever predicting a non-physical attractive force during unloading, which is a known flaw in some potential-based models [17, 22]94.

2.6. How We Solve It: The Numerical Approach

To solve these problems on a computer, we discretize time into small steps. At each time step:

- For the Potential-Based Model: We solve an optimization problem to find the displacement field that minimizes the total energy95.
- For the Non-Potential-Based Model: We solve the nonlinear system of force-balance equations96. This requires more advanced numerical techniques, often relying on fixed-point theorems [10, 13] to guarantee a solution exists97.
- Update the History: After finding the displacement, we update the "high-water mark" for the damage history variable in both models98.

Spatially, we use the well-established Finite Element Method (FEM) to turn the continuous problem into a system of algebraic equations that a computer can solve99.

3. RESULTS

To see how these two different philosophies play out in practice, we ran a series of numerical experiments100. We simulated the response of a cohesive interface to various prescribed loading and unloading paths, using the well-known PPR model [22] as the basis for our loading laws101.

3.1. Case 1: Simple In-Phase Loading

First, we looked at the simplest case: the slips in both modes move in perfect sync, and the fracture energies are the same102.

- Potential-Based Model: During unloading, the force-displacement curve is non-linear103. While this is mathematically consistent, it's often not what is seen in experiments104.
- Non-Potential-Based Model: Here, the unloading path is a straight line back to the origin, by design105. This clean, linear unloading is more typical of how many damage models are formulated106.

3.2. Case 2: More Complex Out-of-Phase Loading

Next, we made things more interesting by having the slips move out of phase107. This creates a "mixed-mode" situation where the interface might be loading in one direction while unloading in another108.

- Potential-Based Model: The model's behavior starts to look strange109. The traction-separation curves form complex, looping paths during unloading that don't represent simple stiffness loss110. The force in one direction becomes oddly dependent on what's happening in the other direction111.
- Non-Potential-Based Model: The response remains clear and predictable. The force in each direction depends only on its own history, leading to the same clean, linear unloading as before112.

3.3. Case 3: The Toughest Test - Unequal Energies and Out-of-Phase Loading

This case is the most challenging. The slips are out of phase, and the fracture energies in the two directions are different—a situation that is very common in real anisotropic materials113.

- Potential-Based Model: The model fails spectacularly. As shown in Figure 2, during a period where the slip in mode 2 is decreasing, the model predicts a negative cohesive force114. This is physically absurd—it suggests that the damaged, separating surfaces are actively pulling themselves back together115. This bizarre result is a direct consequence of the mathematical coupling between the modes in the energy potential116.
- Non-Potential-Based Model: Thanks to our choice of taking only the positive part of the potential's gradient, our model correctly handles this case117. The traction decreases during unloading and goes to zero, as it should, but it never becomes attractive118.

3.4. Case 4: A Special Case - Unidirectional Unloading

Finally, we looked at a very specific case where we unload in one direction while holding the slip in the other direction perfectly still119.

- Potential-Based Model: In this highly controlled scenario, the model behaves reasonably120. The unloading is non-linear, but it doesn't produce the strange loops or negative forces seen in the more general cases121.
- Non-Potential-Based Model: Once again, the model provides a clean, linear unloading response122.

These results paint a clear picture: while the potential-based model works for simple, monotonic loading, it can easily break down and give unphysical results in more realistic, mixed-mode cyclic scenarios123. The non-potential-based framework, however, remains robust and physically consistent throughout124.

4. DISCUSSION

4.1. The Achilles' Heel of Potential-Based Models

Our results confirm what has been suspected for some time: potential-based models have a fundamental weakness when it comes to mixed-mode unloading125. The very thing that makes them elegant—deriving all forces from a single energy function—is also their undoing126. This mathematical structure creates a rigid coupling between the different fracture modes127. While some coupling is needed to capture how a material's toughness might change under different loading directions, it leads to strange predictions during unloading128. When you unload in one direction, the mathematical coupling can cause the force in another direction to change in a way that makes no physical sense, even causing it to become attractive [17]129. What's more, for these models to be consistent, they often require that the fracture energies in all directions be identical [22]130. This is a major limitation, as most real-world anisotropic materials simply don't behave that way131. The non-potential approach neatly sidesteps this problem, allowing for different fracture properties in every direction132.

4.2. The Strength of the Non-Potential Framework: Robustness and Realism

The non-potential-based model, in contrast, proves to be far more flexible and reliable133. By defining the rules of fracture directly, we can build in physically realistic behavior from the start134. Our proposed method—using a linear decay for unloading and taking the positive part of the potential's gradient for loading—solves the key problems of the potential-based approach135. It correctly models stiffness loss and energy dissipation without creating strange cross-effects or non-physical attractive forces136. This has important practical consequences. For any engineering problem that involves fatigue, impact, or complex cyclic loads on composite structures, a non-potential-based CZM is the safer and more accurate choice137. The potential-based approach should be reserved for simpler cases dominated by one-way, monotonic loading, where its mathematical convenience is a real advantage138.

4.3. Where Do We Go From Here? Limitations and Future Directions

While our study provides a rigorous comparison, it's important to acknowledge its limitations139. Our analysis was two-dimensional and assumed a slow, quasi-static process140. The next logical steps are to extend this comparison to full 3D models and to include the effects of dynamics and loading rates [18], which are crucial for modeling things like impacts and vibrations141. Furthermore, the flexibility of non-potential models means they have more parameters that need to be carefully calibrated with experimental data142. A key area for future research is the development of better experimental tests and computational methods to extract these parameters accurately143. Finally, the deep mathematical questions about these complex non-potential systems, such as whether their solutions are always unique, remain an active and challenging area of research [6, 8, 12, 20]144. The ultimate goal is to integrate these highly detailed interface models into larger, multi-scale simulations, perhaps combining them with other methods like phase-field models for bulk fracture [4]145. This would give us a truly powerful predictive tool for ensuring the safety and reliability of complex engineering systems146.

5. CONCLUSION

In this paper, we've carried out a detailed and rigorous comparison of two competing philosophies for modeling fracture: the potential-based (variational) approach and the non-potential-based (non-variational) approach147. Focusing on the problem of sliding elastic laminates under complex loading cycles, we developed a unified framework to construct and analyze both types of models148. Our findings lead to an unambiguous conclusion. Potential-based models, for all their mathematical elegance, are fundamentally ill-suited for capturing the physics of mixed-mode unloading149. They can produce bizarre artifacts, like attractive forces between separating surfaces, and often rely on unrealistic assumptions about material properties150. The non-potential-based framework, on the other hand, by defining the fracture laws directly, offers the flexibility and robustness needed to model complex hysteretic phenomena with high physical fidelity151. This work emphasizes that the choice of a modeling strategy is a critical decision that directly impacts the validity of a simulation152. For problems involving simple, monotonic fracture, potential-based models are an excellent and efficient tool153. However, for the vast and important class of problems involving cyclic loading, fatigue, and complex damage in modern laminated structures, the enhanced realism and reliability of the non-potential-based approach make it the clearly superior and recommended choice154.

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